# STANDARD MAPS AND THE CLASSIFICATION OF TOPOLOGICAL SPACES

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# CHAPTER I

## INTRODUCTION

Let X and Y be topological spaces. In the ensuing discussion we will use the following standard notation: $C(X,Y)$ and $H(X,Y)$ are the collections of continuous surjections and homeomorphisms, respectively. In the event that $X = Y$ we agree to write $C(X)$ and $H(X)$. All function spaces will be endowed with the compact - open topology. In this topology we will write $(F,U)$ for the collection of maps which take F into U, F compact and U open. We use C, $\simeq$ and N to denote the standard middle-thirds Cantor set, "is homeomorphic to", and the positive integers, respectively. Our only nonstandard notation is the letter P for the space of irrationals.

The literature is quite wealthy in results that relate properties of X and $H(X)$. In particular, Whittaker (18) shows that two compact, locally euclidean manifolds are homeomorphic if and only if there is a group isomorphism between their homeomorphism groups. Wechsler (17) obtains the same equivalence for spaces satisfying a strong homogeneity condition. If we set $x' = \{h \in H(X) : h(x) = x\}$, then we obtain a subgroup of $H(X)$. In (11), Mostert gives conditions on X which guarantee that X and $H(X)/x'$ are homeomorphic. A complete listing of the major papers in this category would indeed be lengthy and inappropriate at this time. All of the papers, however, share a common feature. Each imposes topological conditions (usually rather severe) on X which allows an algebraic statement about $H(X)$, or a subgroup of $H(X)$, to take on topological significance. While the three results just mentioned are indeed elegant, their proofs share little common ground and therefore there is room for improvement.

The techniques presented in this paper are certainly not offered as a general panacea for all problems, but rather as a viewpoint which provides a common ground from which to attack. That in itself is useful.

1.0. Definition. Let $f \varepsilon C(X,Y)$ and set $G(f,Y) = \{h \varepsilon H(X) : fh = f\}$. $G(f,Y)$ is a subgroup of $H(X)$. We define $f$ to be a standard map if and only if

i) $f$ is an identification map and

ii) $x,w \varepsilon f^{-1}(y)$ implies there are $x_n$ in $X$ and $h_n$ in $G(f,Y)$ such that $x_n \to x$ and $h_n(x_n) \to w$.

If $X = C$ and $Y$ is compact and metrizable, then Vobach (14) shows that standard maps exist. The existence of such maps provide us with a classification theorem for the class of compact metric spaces. The statement and proof of this theorem will be provided later.

It is easy to show that any locally compact, separable metric space is the continuous image of $N \times C$ where $N$ and $C$ are the positive integers and Cantor set, respectively. It is certainly well known that any complete, separable metric space is the continuous image of $P$, the irrationals. The following questions, in view of (14), are therefore quite naturally posed: Are theorems similar to Vobach's theorem for compact metric spaces available for locally compact (complete) separable metric spaces stated in terms of $H(N \times C)$ ( $H(P)$ )? Part of this research is devoted to providing affirmative answers to these questions.

Having established these theorems one naturally speculates as to whether algebraic or topological properties of G(f,X) determines any topological properties of X. Unfortunately, very little has been established in this direction. However, there are fixed-point theorems and connectivity conditions for X in terms of G(f,X) and some structure theorems for the group G(f,X) itself given in this paper.

In the course of proving these theorems it will be necessary to develope a few of the elementary properties possessed by C and P. No claims of originality are made and because of their "folklore" nature complete proofs are given. We will also develop a few elementary topological properties of standard maps (in the Cantor set - compact metric space setting).

## CHAPTER II

## C AND P

2.0. Proposition. $H(C)$ is totally disconnected.

Proof: Let $\alpha \varepsilon H(C)$ and $D_\alpha$ its component (assume the compact-open topology) in $H(C)$. Let $D_1$ be the component of the identity. $D_1$ is a normal subgroup of $H(C)$ (see (12)). According to a theorem of R.D. Anderson (1) $H(C)$ is simple. Hence $D_1 = \{1_C\}$ or $H(C)$. If $D_1 = \{1_C\}$, then $\alpha^{-1}D_\alpha = \{1_C\}$ which in turn implies $D_\alpha = \{\alpha\}$. If $D_1 = H(C)$, then evidently $H(C)$ is connected. But if $x$ is any fixed point of $C$, the map $p : H(C) \to C$ defined by $p_x(h) = h(x)$ is continuous and onto. Thus, we deduce that $C$ is connected, a contradiction. We conclude that $H(C)$ is totally disconnected.

2.1. Proposition. $H(P)$ is totally disconnected.

Proof: The proof is similar to that of 2.0.

2.2. Theorem. For each $i$ let $A_i = N$. Then $\Pi\{A_i : i \varepsilon N\}$ and $P$ are homeomorphic.

Proof: Let $P' = P \cap [1,\infty)$. Since $P'$ and $P$ are homeomorphic it will suffice to show that $P'$ and $\Pi\{A_i : i \varepsilon N\}$ are homeomorphic. First, let us note that each element $\alpha$ in $P'$ has a unique representation as a continued fraction $\alpha = [\alpha_1, \alpha_2, \ldots]$ where $\alpha_i \varepsilon N$ (see (13)). Topologize $P'$ by

$d(\alpha,\beta) = 1/n$ where n is the first integer such that $\alpha_n \neq \beta_n$ (d is a complete metric and is equivalent to the Euclidean metric).

If $\alpha \in P'$ and $\beta^n \to \alpha$ then $|[\alpha_0, \alpha_1, \ldots] - [\beta_0^n, \beta_1^n, \ldots]| \to 0$, or equivalently, $|(\alpha_0 - \beta_0^n) + (1/[\alpha_1, \ldots] - 1/[\beta_1^n, \ldots])| \to 0$. If for each $n \in N$ there is an $m > n$ such that $\alpha_0 - \beta_0^m \neq 0$, then evidently $1/[\alpha_1, \ldots] - 1/[\beta_1^k, \ldots]$ converges to some integer. However, both fractions are smaller than one and hence their difference does not exceed one. Therefore $\alpha_0 = \beta_0^n$ eventually. It follows by induction that $\alpha_k = \beta_k^n$ eventually for each $k \in N$.

Define $h\colon \Pi\{A_i : i \in N\} \to P'$ by $h((\alpha_1, \alpha_2, \ldots)) = [\alpha_1, \alpha_2, \ldots]$. Clearly h is well-defined, one to one, and onto. The continuity of both h and $h^{-1}$ follows from the discussion in the preceding paragraph.

2.3. Corollary. $C \times P$ and P are homeomorphic.

Proof: For each i in N let $B_i = \{0, 2\}$ with the discrete topology. Let $A_i = N$ for each i in N with the discrete topology. Then $A_i \times B_i$ and $A_i$ are homeomorphic for each i. Therefore $C \times P$ and P are homeomorphic.

2.4. Corollary. Let $P_i = P$ for each i in N. Then $\Pi\{P_i : i \in N\}$ and P are homeomorphic.

Proof: By theorem 2.2 $P_i = \Pi\{A_j^i : j \in N\}$ where $A_j^i = N$ for all i and j in N. Thus $\Pi\{P_i : i \in N\} = \Pi\{\Pi\{A_j^i : j \in N\} : i \in N\} \simeq \Pi\{A_i : i \in N\}$ where $A_i = N$ for each i in N. Therefore $\Pi\{P_i : i \in N\} \simeq P$.

Let $\underline{C}(N,N)$ be the space of maps of N into N endowed with the compact - open topology. Assume that P' has the metric of 2.2.

2.5. Proposition. $\underline{C}(N,N) \simeq P'$.

<u>Proof</u>: Define $\phi$: $\underline{C}(N,N) \to P'$ by $\phi(f) = [f(1), f(2), \ldots]$. Clearly $\phi$ is well-defined and onto.

$\phi$ is continuous. Let $S_{1/n}(\alpha)$ be the open ball in $P'$ at $\alpha$ of radius $1/n$ and $f \in \underline{C}(N,N)$ with $\phi(f) = \alpha$. Define $B_j = (\{j\},\{\alpha_j\}) = \{g \in \underline{C}(N,N)$: $g(j) = \alpha_j\}$ where $\alpha = [\alpha_1, \alpha_2, \ldots]$. Clearly $B_j$ is open in $\underline{C}(N,N)$. Choose $m > n$ and define $B = \cap\{B_j : 1 \le j \le m\}$. Then $B$ is open and $\phi(B) \subset S_{1/n}(\alpha)$.

$\phi$ is one to one: Let $f,g \in \underline{C}(N,N)$ with $f \ne g$. Then there is an $i \in N$ such that $f(i) \ne g(i)$. Thus, $\phi(f) \ne \phi(g)$.

$\phi^{-1}$ is continuous: Let $\alpha \in P'$ and $\alpha^j$ a sequence in $P'$ which converges to $\alpha$. Then $\phi^{-1}(\alpha^j) = (\alpha_1^j, \alpha_2^j, \ldots) \to (\alpha_1, \alpha_2, \ldots) = \phi^{-1}(\alpha)$ where $[\alpha_1^j, \alpha_2^j, \ldots] = \alpha^j$ (Recall that $[\alpha_1^j, \alpha_2^j, \ldots] \to [\alpha_1, \alpha_2, \ldots]$ if and only if $\alpha_i^j \to \alpha_i$). Therefore $\underline{C}(N,N) \simeq P'$.

<u>2.6. Corollary</u>. Let $A_i = N$ for each $i$ in $N$. Then $P' \simeq \Pi\{\underline{C}(A_i, A_,) : i \in N\}$.

<u>Proof</u>: Follows immediately from 2.5 and 2.4

<u>2.7. Theorem</u>. Let $\underline{H}(N,N) = \{h \in \underline{C}(N,N) : h$ is one to one$\}$. Then $P' \simeq \underline{H}(N,N)$.

<u>Proof</u>: Let $\phi : \underline{C}(N,N) \to P'$ be the homeomorphism given in 2.5. Let $P^\# = \{\alpha \in P' : i \ne j \to \alpha_i \ne \alpha_j, \alpha = [\alpha_1, \alpha_2, \ldots]\}$. Clearly $\phi(\underline{H}(N,N)) = P^\#$. Thus, we show $P^\#$ and $P'$ are homeomorphic.

Define $P_k = \{\beta \in P' : i, j \le k, i \ne j$ implies $\beta_i \ne \beta_j\}$. Let $\alpha \in P_k$ and choose $n \in N$ such that $n > k$. Then $\beta \in S_{1/n}(\alpha)$ if and only if $\alpha_j = \beta_j$ for $j$ satisfying $1 \le j \le n$. Therefore $\beta \in P_k$ and so $P_k$ is open in $P'$ for each $k$. Now $P^\# = \cap\{P_k : k \in N\}$ and therefore $P^\#$ is an absolute $G_\delta$.

According to (8) we can conclude that $P'$ and $P^{\#}$ are homeomorphic if no non-empty open subset of $P^{\#}$ is compact. Let $U = P^{\#} \cap V$, $V$ open in $P'$. Let $p \in U$ and choose $n$ such that $S_{1/n}(P) \subset V$ implies $P^{\#} \cap S_{1/n}(P) \subset U$. Define $p^1 = [p_1,p_2,\ldots,p_{n+1},p_{n+3},p_{n+2},p_{n+4},p_{n+5},\ldots]$

$$p^2 = [p_1,p_2,\ldots,p_{n+1},p_{n+4},p_{n+3},p_{n+2},p_{n+5},p_{n+6},\ldots]$$

$$\vdots$$

$$p^k = [p_1,p_2,\ldots,p_{n+1},p_{n+k+2},p_{n+k},p_{n+k-1},\ldots,p_{n+2},p_{n+k+3},\ldots]$$

where $p = [p_1,p_2,\ldots]$. $\{p^j : j \in n\} \subset U$ and has no limit point in $U$. Therefore $U$ is not compact. Hence $P^{\#} \simeq P'$.

2.8. Corollary. Let $A_i = \underline{H}(N,N)$ for each $i$. Then $P' \simeq \{A_i : i \in N\}$.

Proof: Follows from 2.4 and 2.7.

# CHAPTER III

## CLASSIFICATION THEOREMS

This section is devoted to the classification theorems mentioned in the introduction. We will also present the proof of Vobach's fundamental lemma which appears in (14).

Let us recall the definition of a standard map and set $S(X,Y) = \{f \in C(X,Y) : f \text{ is standard}\}$. We shall show for certain choices of X and Y that $S(X,Y) \neq \phi$. The classification theorems will then follow quite easily.

We first prove a stronger version of a proposition by Fort (4).

<u>3.0. Definition</u>. Let F be a set-valued map from a separable metric X to a complete, separable metric space Y. F is upper <u>semi-continuous</u> at x if and only if 1) F(x) is closed in Y and 2) for each neighborhood V of F(x) there is a neighborhood U of x such that $F(U) \subset V$.

<u>3.1. Lemma</u>. Let $\{f_n : n \in N\}$ be a sequence of upper semi-continuous set-valued maps from X to Y, X and Y as in 3.0. If for each $x \in X$ $f_j(x) \subset f_{j+1}(x)$ for all j and diam $f_n(x) \to 0$, then f defined by $f(x) = \cap\{f_n(x) : n \in N\}$ is a continuous function.

<u>Proof</u>: For each k choose $y_k \in f_k(x)$. $\{y_k\}$ is a Cauchy sequence and so has a limit point y. Let U be any neighborhood of y and k any fixed positive integer. Then $U \cap f_k(x) \neq \phi$ and therefore $y \in f_k(x)$.

Therefore $y \in \cap\{f_k(x) : k \in N\}$. If $z \in \cap\{f_k(x) : k \in N\}$, $z \neq y$, then diam $\cap\{f_k(x) : k \in N\} > 0$ which contradicts the hypothesis that diam $f_k(x) \to 0$. Clearly $f(x)$ must be a singleton. Let $U$ be any neighborhood of $f(x)$. If for each $m \in N$ there is an $n \geq m$ such that $f_n(x) \not\subset U$, then there is a $y_n \in f_n(x) - U$ for each $n$ and necessarily $y_n \to f(x)$, a contradiction! Thus, there exists an $m$ such that $n \geq m$ implies $f_n(x) \subset U$. By upper semi-continuity for each $y \geq m$ there is an open set $V_j$ such that $f_j(V_j) \subset U$. Now $f_m(V_m) \supset f_{m+1}(V_m) \supset f_{m+2}(V_m) \supset \ldots$. Hence $f(V_m) \subset U$ and therefore $f$ is continuous at $x$.

We omit the proof of the following standard fact. See, for example, (3).

<u>3.2. Lemma</u>. Let $X$, $Y$ and $Z$ be spaces, $f : X \to Y$ an identification, and $g : X \to Z$ continuous. If $gf^{-1}$ is single-valued, then $gf^{-1}$ is continuous.

<u>3.3. Lemma</u>. Let $X$ be homogeneous and $f \in S(Y,Z)$. Then $f \circ \Pi : X \times Y \to Z$ is standard where $\Pi$ is the projection map on the $Y$ - coordinate.

<u>Proof</u>: Suppose $f(\Pi(x,y)) = f(\Pi(a,b))$. Then $f(y) = f(b)$ and so there are $x_n \in Y$ and $h_n \in G(f,Z)$ such that $x_n \to y$ and $h_n(x_n) \to b$. Choose $g \in H(X)$ with $g(x) = a$. Then $H_n$, defined by $H_n = (g,h_n) \in G(f \circ \Pi, Z)$, $(x,x_n) \to (x,y)$, and $H_n(x,x_n) \to (a,b)$. $f \circ \Pi$ is an identification since $f$ is and $\Pi$ is an open map. Thus, $f \circ \Pi$ is standard.

3.4. Lemma. Let $X$, $Y$ and $Z$ be topological spaces and $f \in S(X,Y)$. Then $Y$ and $Z$ are homeomorphic if and only if there is a $g \in S(X,Z)$ satisfying $G(f,Y) = G(g,Z)$.

Proof: Suppose $h : Y \to Z$ is a homeomorphism. Clearly $h \circ f \in S(X,Z)$ and $G(f,Y) = G(hf,Z)$.

On the other hand, let us assume there is a standard map $g$ of $X$ onto $Z$ satisfying $G(f,Y) = G(g,Z)$. Define $h : Y \to Z$ by $h(y) = gf^{-1}(y)$. We assert that $h$ is single-valued. If $f(a) = f(b)$, then there are $x_n \in X$ and $h_n \in G(f,Y)$ satisfying $x_n \to a$ and $h_n(x_n) \to b$. Therefore $h$ is single-valued. Similarly, $h^{-1}$ is single-valued. By lemma 3.2 both $h$ and $h^{-1}$ are continuous and therefore $h$ is a homeomorphism. This lemma is essentially contained in the main theorem of A. Vobach (14).

At this point our strategy should be clear: Namely, for each class of spaces $\mathcal{S}$ for which there is a 'universal' space $X$ (each element of $\mathcal{S}$ is a continuous image of $X$) we want to establish that $S(X,Y) \neq \phi$ for each $Y \in \mathcal{S}$. By virtue of that fact and lemma 3.4 we will have a classification of all elements in $\mathcal{S}$.

3.5. Lemma. Let $X$ be a complete metric space and for each positive integer $i$ let $A_i \subset N$, $A_i \neq \phi$. If there is a family of closed covers $\mathcal{F}_k = \{F^k(a_1,a_2,\ldots,a_k) : (a_1,a_2,\ldots,a_k) \in A_1 \times A_2 \times \ldots \times A_k\}$ which satisfy the conditions

1. $F^k(a_1,a_2,\ldots,a_k) \supset F^{k+1}(a_1,a_2,\ldots,a_k,n)$ for each $n \in A_{k+1}$,
2. mesh $\mathcal{F}_k \leq 1/k$, and
3. $F^n(a_1,a_2,\ldots,a_n) \supset F^{n+1}(b_1,b_2,\ldots,b_{n+1})$ implies there is a $m \in A_{n+1}$ such that $F^{n+1}(a_1,a_2,\ldots,a_n,m) = F^{n+1}(b_1,b_2,\ldots,b_{n+1})$,

then there is a map $f \in C(A,X)$ (set $A = \Pi\{A_i : i \in N\}$) such that $f(a) = f(b)$ implies there are sequences $a_n \in A$ and $h_n \in G(f,X)$ satisfying $a_n \to a$ and $h_n(a_n) \to b$.

Proof: If $a = (a_1,a_2,\ldots) \varepsilon A$ then set $E_n(a) = E(a_1,\ldots,a_n) = \{b \varepsilon A : b_i = a_i, 1 \leq i \leq n\}$. Let A have the following (Baire type) metric: $d(a,b) = 1/n$ where n is the first integer for which $a_n \neq b_n$. d is a complete metric and agrees with the product topology. Clearly the set $E_n(a)$ is both open and closed for each $a \varepsilon A$ and $n \varepsilon N$.

Define $f : A \to X$ by $f(a) = \cap\{F^n(a_1,\ldots,a_n) : n \varepsilon N\}$. $f \varepsilon C(A,X)$ by lemma 3.1. $f(E_n(a)) = F^n(a_1,\ldots,a_n)$ since on the one hand $b \varepsilon E_n(a)$ implies $f(b) = \cap\{F^k(b_1,b_2,\ldots,b_k) : k \varepsilon N\} \subset F^n(a_1,\ldots,a_n)$. On the other hand, if $x \varepsilon F^n(a_1,\ldots,a_n)$, then there is a sequence $\{a_j' : a_j' \varepsilon A_j, n + 1 \leq j < \infty\}$ such that $x = \cap\{F^{n+k}(a_1,\ldots,a_n,a_{n+1}',\ldots a_{n+k}') : k \varepsilon N\} = f(b)$ where $b = (a_1,a_2,\ldots,a_n,a_{n+1}',a_{n+2}',\ldots)$. Clearly $b \varepsilon E_n(a)$.

Now, let us suppose that $a \neq b$, $f(a) = f(b)$, and let $\varepsilon > 0$ be given. Choose n such that $E_n(a) \cap E_n(b) = \phi$ and $1/n < \varepsilon$. Since $f(E_n(a)) = F^n(a_1,\ldots,a_n)$, $f(E_n(b)) = F^n(b_1,b_2,\ldots,b_n)$ and both contain $f(a) = f(b)$, there is a $F^{n+1}(c_1,\ldots,c_{n+1})$, $a'_{n+1} \varepsilon A_{n+1}$ and $b'_{n+1} \varepsilon A_{n+1}$, such that

$f(a) = f(b) \varepsilon F^{n+1}(c_1,\ldots,c_{n+1}) = F^{n+1}(a_1,\ldots,a_n,a_{n+1}') =$
$F^{n+1}(b_1,\ldots,b_n,b_{n+1}')$.

Let $\alpha \varepsilon E(a_1,\ldots,a_n,a_{n+1}')$. $d(\alpha,a) < 1/n < \varepsilon$ as required and we must now exhibit $h \varepsilon G(f,X)$ such that $d(h(\alpha),b) < \varepsilon$. We define h in terms of a sequence of set-valued maps. On $A - \{E^{n+1}(a_1,\ldots,a_n,a_{n+1}') \cup E^{n+1}(b_1,\ldots b_n,b_{n+1}')\}$ let h be the identity; on $E^{n+1}(a_1,\ldots,a_n,a_{n+1}') \cup E^{n+1}(b_1,\ldots,b_n,b_{n+1}')$ we use a sequence of set-valued maps. Set

$$H_1(d) = \begin{cases} E^{n+1}(a_1,\ldots,a_n,a_{n+1}') \text{ if } d \varepsilon E^{n+1}(b_1,\ldots,b_n,b_{n+1}') \\ E^{n+1}(b_1,\ldots,b_n,b_{n+1}') \text{ if } d \varepsilon E^{n+1}(a_1,\ldots,a_n,a_{n+1}') \end{cases}$$

Now $E^{n+1}(a_1,\ldots,a_n,a_{n+1}') = \cup\{E^{n+2}(a_1,\ldots,a_n,a_{n+1}',a_{n+2}') : a_{n+2}' \varepsilon A_{n+2}\}$ and $E^{n+2}(b_1,\ldots,b_n,b_{n+1}') = \cup\{E^{n+2}(b_1,\ldots,b_n,b_{n+1}',b_{n+2}') : b_{n+2}' \varepsilon A_{n+2}\}$. Moreover, for some choice of subscripts in the $(n + 2)$th coordinate,

$f(E^{n+2}(a_1,\ldots,a_n,a'_{n+1},a'_{n+2})) = f(E^{n+2}(b_1,\ldots,b_n,b'_{n+1},b'_{n+2}))$. Let's assume, as indicated, that the choice is $a'_{n+2}$ and $b'_{n+2}$. No generality is lost by this assumption. Set

$$H_2(d) = \begin{cases} E^{n+2}(a_1,\ldots,a_n,a'_{n+1},a'_{n+2}) \text{ if } d\varepsilon E^{n+2}(b_1,\ldots,b_n,b'_{n+1},b'_{n+2}) \\ E^{n+2}(b_1,\ldots,b_n,b'_{n+1},b'_{n+2}) \text{ if } d\varepsilon E^{n+2}(a_1,\ldots,a_n,a'_{n+1},a_{n+2}) \end{cases}$$

for each such pair $a'_{n+2},b'_{n+2}$ in $A_{n+2}$. In general, set

$$H_k(d) = \begin{cases} E^{n+k}(a_1,\ldots,a_n,a'_{n+1},\ldots,a'_{n+k}) \text{ if } d\varepsilon E^{n+k}(b_1,\ldots,b_n,b'_{n+1},\ldots,b'_{n+k}) \\ E^{n+k}(b_1,\ldots,b_n,b'_{n+1},\ldots,b'_{n+k}) \text{ if } d\varepsilon E^{n+k}(a_1,\ldots,a_n,a'_{n+1},\ldots,a'_{n+k}) \end{cases}$$

where the above pairs of sets are chosen so that they have identical f - images.

Now we define h on $E^{n+1}(a_1,\ldots,a_n,a'_{n+1}) \cup E^{n+1}(b_1,\ldots,b_n,b'_{n+1})$ by $h(d) = \cap\{H_k(d) : k \in N\}$. By lemma 3.1 h is continuous and onto. But $h^2 = 1_A$ and therefore $h^{-1}$ is continuous and onto. $h \in G(f,X)$ since it fixes each point or switches it with a point in the same pre-image. Finally, $d(h(\alpha),b) < 1/n < \varepsilon$.

Remark: We should note at this point that lemma 3.5 does not establish that f is standard.

The next lemma is a generalization of a useful fact established by Vobach in (15).

3.6. Lemma. If $f \in S(X,Y)$ and h is a homeomorphism from Z to X, then fh is an element of $S(Z,Y)$.

Proof: Let h be a homeomorphism between X and Z. If $f(h(a)) = f(h(b))$, then there is a sequence $x_n$ in X and $h_n \in G(f,Y)$ such that $x_n \to h(a)$ and $h_n(x_n) \to h(b)$. Define $\alpha_n$ in $H(Z)$ by $\alpha_n = h^{-1}\circ h_n \circ h$. $f\circ h\circ\alpha_n = f\circ h\circ h^{-1}\circ h_n\circ h = f\circ h$. Thus, $\alpha_n \in G(f\circ h,Y)$. Set $z_n = h^{-1}(x_n)$. Then

$z_n \to a$ and $\alpha_n(z_n) = h^{-1}(h_n(h(h^{-1}(x_n)))) = h^{-1}(h_n(x_n)) \to h^{-1}(h(b)) = b$. Therefore $f \circ h \in S(Z,Y)$

The next result appears in (14). It is the main conclusion of Vobach's paper. It is mentioned here only because we obtain it so easily from lemma 3.5.

3.7. Theorem. If $X$ is compact and metrizable, then $S(C,X) \neq \phi$.

Proof: Let $\{T(j) : 1 \leq j \leq n_1\}$ be a finite closed cover of $X$ whose mesh does not exceed 1. For each $T(k)$ choose a finite number of closed subsets $T(k,l)$ of diameter less than or equal to 1/2 whose union is $T(k)$. Label this collection $\{T(k,l) : 1 \leq l \leq k_2\}$. Now consider each $T(k) \cap T(m)$ which is non-empty and write it as a finite union of closed subsets none of whose diameter exceeds 1/2. Label each set in this union in two ways: $T(k,l)$ and $T(m,l)$. We now add each set to the collection already obtained for $T(k)$ and $T(m)$. We now have the closed cover $\{T(k,l) : 1 \leq k \leq n_1$ and $1 \leq l \leq k_2'\}$ where $k_2'$ is some integer larger than $k_2$ which accommodates the sets obtained by considering the intersections. Define $n_2 = \text{maximum } \{k_2'\}$ and go back to each collection $\{T(k,l) : k \text{ fixed}, 1 \leq l \leq k_2'\}$ and add a sufficient number of sets (diameter $\leq 1/2$) to obtain a total of $n_2$ sets. The induction is clear. To establish our assertion we need only to note that the family of closed covers $\{T(a_1,\ldots,a_n) : 1 \leq a_i \leq n_i, n \in N\}$ satisfies the conditions of lemma 3.5. Each $A_i$ (see proof of 3.5.) is finite and therefore $A$ is a Cantor set. The proof is completed by observing that $f$ is an identification since $f$ is closed.

3.8. Theorem. (Vobach (14). Let X and Y be compact metric spaces and f a standard map of C onto X. X and Y are homeomorphic if and only if there is a standard map g of C onto Y satisfying $G(f,X) = G(g,Y)$.

Proof: Lemma 3.4.

Our next task is to extend the results just obtained, namely 3.7 and 3.8, to the class of locally compact, separable metric spaces.

Notation: Let us denote $C - \{1\}$ by $C'$. Since C is homogeneous it is clear that $C - \{\alpha\} \simeq C'$ for all $\alpha \in C$.

3.9. Lemma. $C' \simeq C \times N$.

Proof: Choose a pairwise disjoint collection $\{C_k : k \in N\}$ of open Cantor subsets of C such that $C' = \cup\{C_k : k \in N\}$. For each $n \in N$ let $h_n : C \times \{n\} \to C_n$ be a homeomorphism. Define $h : C \times N \to C'$ by $h(c,n) = h_n(c)$. Clearly h is well-defined, one-to-one and onto. Let U be an open subset of C'. Then $U = \cup\{C_k \cap U : k \in N\}$. We have $h^{-1}(U) = \cup\{h^{-1}(C_k \cap U) : k \in N\} = \cup\{h_k^{-1}(C_k \cap U) : k \in N\}$ which is open in C x N. Accordingly, h is continuous. Let V be an open subset of C x N. Then $V = \cup\{V_n \times \{k_n\} : V_n$ is open in C, $k_n \in N, n \in N\}$ and $h(V) = \cup\{h(V_n \times \{k_n\}) : n \in N = \cup\{h_{k_n}(V_n) : n \in N\}$ which is certainly open in C' since each $C_n$ is an open subset of C. Therefore h is a homeomorphism.

3.10. Theorem. If M is a locally compact separable metric space, then $S(C \times N, M) \neq \phi$.

Proof: In view of lemmas 3.6 and 3.9 it is sufficient to show $S(C',M) \neq \phi$. Let $M^*$ be the one-point conpactification of M. $M^*$ is a compact metric space. Construct the sequence of closed covers required by lemma 3.5 in such a way that the ideal point p belongs to exactly one element of each cover in the sequence. Let f be the element of $S(C,M^*)$ so induced. We assert that $f^{-1}(p)$ is a singleton. If $f(a) = f(b) = p$, then evidently p must belong to more than one element of some cover, contrary to hypothesis. Hence, $f^{-1}(p)$ is a singleton as claimed.

Define g from $C' = C - f^{-1}(p)$ onto M by $g(c) = f(c)$. Since each element of $G(f,M^*)$ fixes $f^{-1}(p)$ we have the inclusion $G(f,M^*) \subset G(g,M)$, $h \in G(f,M^*)$ restricted to C' of course.

g is an identification: Let U be a subset of M such that $g^{-1}(U)$ is open in C'. Since $p \notin U$, $f^{-1}(U) = g^{-1}(U)$. C' is open in C and therefore $f^{-1}(U)$ is open in C. Therefore, U is open in $M^*$ and must be open in M since $p \notin U$. Hence g is an identification.

<u>3.11. Theorem.</u> Let X and Y be locally compact separable metric spaces and $f \in S(C \times N, X)$. X and Y are homeomorphic if and only if there is a standard map g of $C \times N$ onto Y satisfying $G(f,X) = G(g,Y)$.

<u>Proof</u>: Lemma 3.4.

<u>3.12. Corollary</u>. If X is a locally compact separable metric space, then $S(P,X) \neq \phi$.

<u>Proof</u>: This follows immediately from 2.2, 2.3, 3.3, and 3.10.

<u>3.13. Theorem.</u> Let X and Y be locally compact separable metric spaces and $f \in S(P,X)$. X and Y are homeomorphic if and only if there is a $g \in S(P,Y)$ satisfying $G(g,Y) = G(f,Y)$.

<u>Proof</u>: Lemma 3.4.

3.14. Corollary. Let $f$ and $g$ be standard maps of $C \times N$ $(P)$ onto locally compact separable metric spaces $X$ and $Y$. $X$ and $Y$ are homeomorphic if $G(f,X)$ and $G(g,Y)$ are conjugate in $H(C \times N)$ $(H(P))$.

Proof: The proof of a similar corollary in (14) is utilized here. We establish the result for $P$ only since the other case is essentially the same.

Since $G(f,X)$ and $G(g,Y)$ are conjugate in $H(P)$ there is an $h \in H(P)$ such that $G(f,X) = h^{-1}G(g,Y)h$. Define $p \in S(P,X)$ by $p = f \circ h^{-1}$. We assert that $G(f \circ h^{-1},X) = G(g,Y)$. If $\alpha \in G(g,Y)$, then $h^{-1} \circ \alpha \circ h \in G(f,X)$ and so $f \circ h^{-1} \circ \alpha \circ h = f$, or $f \circ h^{-1} \circ \alpha = f \circ h^{-1}$. Hence $\alpha \in G(f \circ h^{-1}, X)$. Conversely, if $f \circ h^{-1} \circ \beta = fh^{-1}$ then $f \circ h^{-1} \beta \circ h = f \circ h^{-1} \circ h = f$. Hence $h^{-1} \circ \beta \circ h \in G(f,X)$, or $\beta \in G(g,Y)$. The proof now follows from 3.13.

We have now established classification theorems (and assorted corollarys) for the class of locally compact, separable metric spaces as well as for the class of compact metric spaces. Our last effort in this direction is to establish essentially the same set of theorems for the complete, separable metric spaces.

3.15. Theorem. If $X$ is complete, separable and metrizable, then $S(P,X) \neq \phi$.

Proof: Let us first construct a sequence of closed covers which satisfy the three conditions set forth in lemma 3.5. Let $\{F(k) : k \in N\}$ be a closed cover of $X$ of mesh one or less. For each $F(k)$ choose closed subsets $\{F(k,2n) : n \in N\}$ of diameter less than or equal to 1/2 whose union is $F(k)$. For each non-empty intersection $F(k) \cap F(n)$ choose closed subsets $\{F(k,4m-1) = F(n,4m-1) : m \in N\}$ of diameter no more than 1/2 whose union is $F(k) \cap F(n)$. We now reconsider each collection so formed for each k and add a sufficient number of subsets of diameter less than or equal to 1/2 to insure that each element of N appears as a second coordinate for each k. The induction is clear. The family $\{F(k_1,\ldots,k_n) : k_i \in N, n \in N\}$ satisfies the hypothesis of lemma 3.5. Since each $A_i = Z$ (see proof of 3.5.) we have a map $f : P \to X$ such that $a \neq b$, $f(a) = f(b)$ implies there are $a_n \in X$ and $h_n \in G(f,X)$ such that $a_n \to a$ and $h_n(a_n) \to b$. It remains for us to show that f is an identification. Unfortunately, this may not be true. However, what we can do is construct a second map from P to X by using the map f in such a way that the new map retains the "transitive point-inverses" as well as being an identification.

In (10), Michaels and Stone prove that any metric space which is the continuous image of the irrationals is also a quotient of the irrationals. In proving this result they construct a sequence $\{Z_i : i \in N\}$ of spaces, each homeomorphic to P (let $h_i : Z_i \to P$ be the homeomorphism), such that (1) $Z = \cup\{Z_i : i \in N\}$ and P are homeomorphic and (2) $g : Z \to X$ defined by $g(z) = f(h_i(z))$ if $z \in Z_i$ is an identification, where f is a continuous map of P onto X. For our purposes, f is the map constructed at the beginning of the proof. We intend to show that g is standard. For each $\alpha \in G(f,X)$ define $\alpha_{ij}$ in H(Z) by

$$\alpha_{ij}(z) = \begin{cases} h_j^{-1}\alpha h_i(z) & z \in Z_i \\ h_j^{-1}\alpha h_j(z) & z \in Z_j \\ z & z \notin Z_i \cup Z_j \end{cases}$$

First let us note that $\alpha_{ij} \in G(g,X)$. Let $z \in Z$ and consider $g(\alpha_{ij}(z))$. If $z \notin Z_i \cup Z_j$, then clearly $g(\alpha_{ij}(z)) = g(z)$. If $z \in Z_i$, then $g(\alpha_{ij}(z)) = f(h_j(h_j^{-1}(\alpha(h_i(z))))) = f(h_i)z)) = g(z)$. Similarly for $z \in Z_j$.

Now, let $a \neq b$, $g(a) = g(b)$. Then $f(h_i(a)) = f(h_j(b))$ if $a \in Z_i$,

$b \in Z_j$. Therefore there are sequences $p_n \in p$ and $\alpha^n \in G(f,X)$ such that $p_n \to h_i(a)$ and $\alpha^n(p_n) \to h_j(b)$. Consider $h_i^{-1}(p_n)$. $h_i^{-1}(p_n) \to a$ and $\alpha_{ij}^n(h_i^{-1}(p_n)) = h_j^{-1}(\alpha_n(h_i(h_i^{-1}(p_n)))) = h_j^{-1}(\alpha_n(p_n)) \to h_j^{-1}(h_j(b)) = b$. Thus, g is a standard map of Z onto X.

3.16. Theorem. Let X and Y be complete separable metric spaces and $f \in S(P,X)$. X and Y are homeomorphic if and only if there is a standard map g of P onto Y satisfying $G(f,X) = G(g,Y)$.

Proof: Lemma 3.4.

3.17. Corollary. Let f and g be standard maps of P onto complete separable metric spaces X and Y. If $G(f,X)$ and $G(g,Y)$ are conjugate in $H(P)$, then X and Y are homeomorphic.

Proof: See the proof of 3.14.

# CHAPTER IV

## SOME STRUCTURE THEOREMS FOR G(f,X) AND REMARKS CONCERNING EQUIVALENCE RELATIONS

4.0. Definition. Let X be a compact metric space and G a subgroup of H(X). Define the relation $\sim$ as follows: $x \sim y$ if and only if there are sequences $x_n \in X$ and $h_n \in G$ such that $x_n \to x$ and $h_n(x_n) \to y$.

We have already seen how each compact metric space determines a group which in turn classifies that space. It is therefore reasonable to ask the following question: Let G and $\sim$ be as in 4.0. If $\sim$ is an equivalence relation, is $X/\sim$ metrizable? Such groups obviously exist in view of 3.7. Consider any G(f,Y) where $f \in S(C,Y)$.

Remark: For all groups $G \subset H(X)$ $\sim$ is reflexive and symmetric.

4.1. Theorem. Define $x\ R_2 y$ ($x\ R_3\ y$) with respect to $G \subset H(X)$ if and only if for each $\varepsilon > 0$ there is a $c \in X$ and $h \in G$ such that $d(x,o(c)) + d(o(c),y) < \varepsilon$ ($d(x,F(c)) + d(F(c),y) < \varepsilon$) where d, o(c) and F(c) are a metric for X, the orbit of c, and the orbit closure of c, respectively. For the moment let $\sim = R_1$ (see 4.0.). If $R_i$ is transitive, then $R_j$ and $R_k$ are transitive, $i,j,k \in \{1,2,3\}$. Moreover, all three relations determine precisely the same set of equivalence classes.

Proof: Clearly $R_1$ transitive implies that $R_2$ and $R_3$ are transitive. Suppose $x\ R_i\ y$, $y\ R_i\ z$ for i = 1, 2, 3 and $R_3$ is transitive. Given

$\varepsilon > 0$ there is a $c \in X$ such that $d(x,F(c)) + d(F(c),z) < \varepsilon/4$. Therefore there are $h$ and $k$ in $G$ satisfying $d(x,h(c)) + d(k(c),z) < \varepsilon$. Hence, $x\ R_2\ z$. We also have $d(x,h(c)) + d(k(h^{-1}(h(c))),z) < \varepsilon$ and so $x\ R_1\ z$. Thus, $R_1$ and $R_2$ are transitive. In a similar way we can show the other possibility as well as the last statement of the proposition.

4.2. Definition. Let $\sim$ be as in 4.0. Set $V(X) = \{G \subset H(X) : \sim$ is transitive and $G$ is a group$\}$. For the remainder of this section $[x]_G$, or simply $[x]$ if $G$ is understood, will denote the equivalence class containing $x$.

4.3. Definition. Let $F \subset X$. $F^* = \cup\{[x]_G : [x]_G \cap F \neq \phi\}$.

4.4. Lemma. If $G \in V(X)$, then $F^*$ is closed for each closed subset $F$ of $X$.

Proof: Let $F$ be a closed subset of $X$ and $w \in \overline{F^*}$. Then there is a sequence $\{u_n : n \in N\}$ in $F^*$ which converges to $w$. Let $x_n$ be that element of $F$ such that $u_n \sim x_n$. Without loss of generality we assume that $x_n$ converges to some $x \in F$. For each $n \in N$ there is a $z_n$ in $X$ and $h_n$ in $G$ satisfying $d(u_n,z_n) + d(h_n(z_n),x_n) < 1/n$. But $d(w,z_n) + d(h_n(z_n),x) \leq d(w,u_n) + d(u_n,z_n) + d(h_n(z_n),x_n) + d(x_n,x)$ and $d(w,u_n) + d(x_n,x) \to 0$ as $n \to \infty$. Therefore $w \sim x$ and $w \in F^*$. Hence $F^*$ is closed.

Notation: Let $X/\sim = \{[x]_G : x \in X\}$.

4.5. Theorem. If $G \in V(X)$ and $X/\sim$ has the quotient topology, then $X/\sim$ is compact and metrizable.

Proof: By lemma 4.4. $\{[x]_G : x \in X\}$ is an upper semi-continuous decomposition of $X$. Since $X$ is compact, a well-known theorem (see (9)) for example) gives us that $X/\sim$ is Hausdorff. Hence $X/\sim$ is metrizable.

We have presented a collection of subgroups whose associated quotient spaces are metrizable. Namely, the collection $V(X)$. An interesting question therefore is: If $G$ is a subgroup of $H(X)$, what are sufficient (algebraic or topological) conditions that insure $G \varepsilon V(X)$? Unfortunately, no substantial progress has been made in this direction. There are sufficient conditions known which force $G \varepsilon V(X)$, but as we shall see, they are far too severe to be of any value.

4.6. Definition. (Gottschalk and Hedlund (6)). Let $T$ be a topological group. $A \subset T$ is left-syndetic if there is a compact $K \subset T$ satisfying $KA = T$.

4.7. Definition. (Gottschalk and Hedlund (6)). Let $T$ be a subgroup of $H(X)$ and assume $H(X)$ is a topological group. $T$ is almost periodic at $x$ if and only if for each neighborhood $U$ of $x$ there is a left-syndetic subset $A$ of $T$ such that $xA \subset U$.

4.8. Theorem. (Gottschalk and Hedlund (6)). Let $X$ be a compact Hausdorff space. If $T$ is almost periodic at $x$ for each $x$ in $X$ if and only if the class of orbit-closures is an upper semi-continuous continuous collection.

4.9. Theorem. Let $X$ be a compact metric space of positive dimension. There does not exist a map $f \varepsilon C(C,X)$ such that $\overline{cG(f,X)} \overset{\text{def}}{=} \overline{\{\alpha(c) : \alpha \varepsilon G(f,X)\}} = f^{-1}(f(c))$ for each $c \varepsilon C$.

Proof: Suppose such a map $f$ does exist. Since $C$ is compact and $X$ is metrizable, certainly $f$ is closed. On the other hand, $\{\overline{cG(f,X)} : c \varepsilon C\}$

decomposes C and hence the projection map $p : C \to C/\sim$ ($x \sim y$ iff $x$ and $y$ belong to the same orbit closure) is open (see (6), p. 7). We have the following commutative diagram

$$\begin{tikzcd} C \arrow[rr, "p"] \arrow[dr, "f"'] & & C/\sim \\ & X \arrow[ur, "h = f \circ p^{-1}"'] & \end{tikzcd}$$

where $h$ is a homeomorphism defined by $h \circ p = f$. Hence $f$ is open and $X$ must be zero dimensional, a contradiction.

<u>4.10. Corollary</u>. Let $X$ be a compact metric space with positive dimension. If $f$ is a standard map of $C$ onto $X$, then $G(f,X)$ is not almost periodic.

<u>Proof</u>: If $G(f,X)$ is almost periodic then we claim $f^{-1}(f(x)) = \overline{xG(f,X)}$. Surely we have $\overline{xG(f,X)} \subset f^{-1}(f(x))$. If $y \in f^{-1}(f(x))$ and $U_n$ is a neighborhood base at $x$, then $y \in \cap\{\overline{U_n G} : n \in N\}$ by the standardness of $f$. But, a well-known theorem (2.31 (6)) gives $\cap\{\overline{U_n G(f,X)} : n \in N\} = \overline{xG(f,X)}$. Therefore $\{\overline{cG(f,X)} : c \in C\}$ is a decomposition of $C$. However, this implies (4.10 (6)) that $G(f,X)$ is almost periodic, contrary to theorem 4.9.

<u>4.11. Theorem</u>. Let $X$ be a locally compact, separable metric space. If $f \in C(N \times C, X)$ and $\overline{zG(f,X)} = f^{-1}(f(z))$ for each $z \in N \times C$, then $X$ has dimension zero.

<u>Proof</u>: Let $p : N \times C \to N \times C/\sim$ be the projection map. Then $p$ is both open and closed (see 1.41, 4.10, 4.17, and 4.18 in (6)). We have

$$\begin{tikzcd} N \times C \arrow[rr, "p"] \arrow[dr, "f"'] & & N \times C/\sim \arrow[dl, "h = f \circ p^{-1}"] \\ & X & \end{tikzcd}$$

where h is a homeomorphism defined by $h \circ p = f$. Therefore f is both open and closed. Hence $\dim X = \dim (N \times C) = 0$.

<u>Notation</u>: If $G \subset H(X)$, let $X/G$ be the orbit space with the quotient topology.

<u>4.12. Corollary</u>. Let X be a compact space of positive dimension. If $f \in S(C,X)$ then $C/G(f,X)$ is not $T_1$.

<u>Proof</u>: Consider the following diagram:

$$\begin{tikzcd} C \arrow[rr, "p"] \arrow[dr, "f"'] & & C/G(f,X) \arrow[dl, "h"] \\ & X & \end{tikzcd}$$

where h is a homeomorphism defined by $hp = f$. If $X/G(f,X)$ is $T_1$ then $cG(f,X) = \{\alpha(c) : \alpha \in G(f,X)\} = \overline{cG(f,X)}$ partitions C and so $G(f,X)$ is almost periodic (see 4.10. (6)), contrary to the conclusions of 4.10.

Let X be a compact metric space and G a subgroup of $H(X)$. Let us assume that G is almost periodic so that $X/\sim$ ($x \sim y$ iff x and y belong to the same orbit closure) is metrizable. Let $p : X \to X/\sim$ be the projection map. Surely p is a standard map. The price we must pay is $\dim X/\sim = 0$ which somewhat limits the scope of our study.

There are some algebraic statements that can be made about $G(f,X)$.

They are completely structural in nature and have little bearing on $X$ since virtually all $G(f,X)$ must possess these properties.

4.13. Theorem. Let $M$ be a compact metric space not homeomorphic to the Cantor set $C$. If $f \in S(C,M)$, then

(1) $G(f,M)$ is not free, and

(2) $G(f,M)$ is centerless.

Proof: (1). Since $f$ is not one-to-one $G(f,M) \neq \{1_C\}$. The proof of 3.5 allows us to construct $h \in G(f,M)$ such that $h^2 = 1_C$.

(2). Choose $m \in M$ such that $f^{-1}(m)$ is not a singleton. There is an $x \in f^{-1}(m)$ such that $h(x) \neq x$, $h \in G(f,M)$. Using the technique in 3.5 we can construct $\alpha \in G(f,M)$ such that $\alpha(x) = x$, $\alpha(h(x)) \neq h(x)$. Then $h(\alpha(x)) = h(x) \neq \alpha(h(x))$. Thus, $G(f,M)$ is centerless.

The next theorem is a generalization of a result by Vobach (15).

4.14. Theorem. Let $D$ be a compact homogeneous metric space such that $\Pi\{D_i : D_i = D \text{ for each } i \in N\}$ and $D$ are homeomorphic. Let $\{X_i : i \in N\}$ be a collection of Hausdorff spaces for which $S(D_i,X_i) \neq \phi$ for each $i \in N$. Set $X = \Pi\{X_i : i \in N\}$. Then there are $p \in S(D,X)$ and $p_i \in S(D,X_i)$ satisfying $G(p,X) = \cap\{G(p_i,X_i) : i \in N\}$.

Proof: Let $h \in H(D, \Pi\{D_i : i \in N\})$ and $q_i \in S(D_i,X_i)$. Define $p_i \in S(D,X_i)$ by $p_i = q_i \circ \Pi_i \circ h$ where $\Pi_i : \Pi\{D_i : i \in N\} \to D_i$ is the projection map on the $i^{\underline{th}}$ coordinate. By successive applications of 3.3 and 3.6 $p_i$ is standard for each $i$. Define $p : D \to X$ by $p(x) = \{p_i(x) : i \in N\}$. Note that $p = \{q_i \circ \Pi_i \circ h : i \in N\}$ is equivalent to $h$ followed by $q = \{q_i : i \in N\} : \Pi\{D_i : i \in N\} \to \Pi\{X_i : i \in N\}$. Accordingly, we show $q$ is standard. Suppose $\{q_i(x_i) : i \in N\} = \{q_i(y_i) : i \in N\}$. Then $q_i(x_i) = q_i(y_i)$ for each $i$ and so there are sequences $z_i^j$ in $D_i$ and $\alpha_i^j$ in

$G(q_i,X_i)$ satisfying $z_i^j \underset{j}{\to} x_i$ and $\alpha_i^j(z_i^j) \underset{j}{\to} y_i$. Define the sequence $\{z_j : j \in N\}$ in $\Pi\{D_i : i \in N\}$ by $z_j = \{z_i^j : i \in N\}$. Clearly $z_j \to \{x_i : i \in N\}$. Let $h_j = \{\alpha_i^j : i \in N\}$. Then $h_j(z_j) = \{\alpha_i^j(z_i^j) : i \in N\} \underset{j}{\to} \{y_i : i \in N\}$. Furthermore, $gh_i = \{q_i : i \in N\} \circ h_i = \{q_i \circ \alpha_i^j : i \in N\} = \{q_i : i \in N\} = q$ as required. Clearly $q$ is an identification.

If $\alpha \in G(p,X)$, then $p(\alpha(x)) = \{p_i(\alpha(x)) : i \in N\} = \{p_i(x) : i \in N\}$ which implies $p_i(\alpha(x)) = p_i(x)$ for all $i$. Hence $\alpha \in \cap\{G(p_i,X_i) : i \in N\}$. Similarly, $\alpha \in G(p_i,X_i)$ for each $i \in N$ gives $p_i \circ \alpha = p_i$, or equivalently, $p \circ \alpha = p$. Hence $\alpha \in G(p,X)$.

4.15. Definition. (See (2)). Let $G$, $\{G_i : i \in I\}$ be groups such that $G = \oplus_i G_i$. Define $\Pi_i : G \to G_i$ by $\Pi_i\{g_i : i \in I\} = g_i$. A subgroup $H$ of $G$ such that $\Pi_i(H) = G_i$ for each $i \in I$ is called a subdirect sum of the $G_i$.

A well-known theorem in group theory (see (2)) is the following.

4.16. Theorem. Let $G$ be a group with normal subgroups $\{G_i : i \in I\}$ satisfying $\cap\{G_i : i \in K \subset I\} = \{1_G\}$. Then $G$ is isomorphic to a subdirect sum of the groups $\{G/G_i : i \in I\}$.

Let $M$ be a compact metric space and $p \in S(C,M)$. If $H \subset M$, let $S(p,H) = \{h \in G(p,M) : h(x) = x \text{ for each } x \in C - p^{-1}(H)\}$. According to (16), $S(p,H)$ is a normal subgroup of $G(p,M)$. Again, we generalize a result of Vobach's (16).

4.17. Theorem. Let $M$ be a compact metric space and $p \in S(C,M)$. If $M = \cup\{H_\alpha : \alpha \in A\}$, then $G(p,M)$ is isomorphic to a subdirect product of $\{G(p,M) / S(p,H_\alpha - \cup\{H_\beta : \beta \neq \alpha\})$.

Proof: We need only to show that $\{S(p,H_\alpha - \cup\{H_\beta : \beta \neq \alpha\})\}$ is a disjoint collection of normal subgroups of $G(p,M)$. Suppose $\beta \neq \sigma$ and let $h \in S(p,H_\beta - \cup\{H_\alpha : \alpha \neq \beta\}) \cap S(p,H_\sigma - \cup\{H_\alpha : \alpha \neq \sigma\})$. Then $h(x) = x$ for each $x$ in $\{C - p^{-1}(H_\beta - \cup\{H_\alpha : \alpha \neq \beta\}) \cup \{C - p^{-1}(H_\sigma - \cup\{H_\alpha : \alpha \neq \sigma\} = p^{-1}(\cup\{H_\alpha : \alpha \neq \beta\}) \cup p^{-1}(\cup\{H_\alpha : \alpha \neq \sigma\}) = C$. Therefore $h = 1_C$. The conclusion follows from 4.16.

Let $X$ be a compact metric space and $f \in S(C,X)$. The last results in this section deal with the relationship between $G(f,X)$ and $X/F$, $F$ a non-empty closed-open subset of $X$.

Let $X$, $f$ and $F$ be as in the preceding paragraph. Then $f^{-1}(F)$ is a Cantor subset of $C$. For each $\alpha \in H(f^{-1}(F))$ define $\overline{\alpha}$ in $H(C)$ by

$$\overline{\alpha}(c) = \begin{cases} c & c \notin f^{-1}(F) \\ \alpha(c) & c \in f^{-1}(F) \end{cases}$$

Define $\mathcal{F} = \{\overline{\alpha} \in H(C) : \alpha \in H(f^{-1}(F))\}$. $\mathcal{F}$ is a subgroup of $H(C)$.

For each $\alpha \in G(f,X)$ define $\alpha^* \in G(f,X)$ by

$$\alpha^*(c) = \begin{cases} c & c \in f^{-1}(F) \\ \alpha(c) & c \notin f^{-1}(F) \end{cases}$$

Define $G^* = \{\alpha^* : \alpha \in G(f,X)\}$. Clearly $G^*$ is a subgroup of $G(f,X)$.

<u>4.18. Theorem</u>. The following statements are true.

1) $\mathcal{F}$ is a subgroup of $H(C)$,

2) $G^*$ is a normal subgroup of $G(f,X)$, and

3) $G^*$ is a continuous homomorphic image of $G(f,X)$.

<u>Proof</u>: 1) is obvious. 2) Let $\alpha \in G^*$, $h \in G(f,X)$, and $x \in f^{-1}(F)$. Then $h(x) \in f^{-1}(F)$ and $\alpha(h(x)) = h(x)$. Thus, $h^{-1}(\alpha(h(x))) = x$. Hence $(h^{-1}\circ\alpha\circ h)^* = h^{-1}\circ\alpha\circ h$ and $h^{-1}\circ\alpha\circ h \in G^*$. For $\alpha$, $\beta \in G^*$ we surely have $(\alpha\circ\beta^{-1})^* = \alpha\circ\beta^{-1}$. 3) Define $\theta : G(f,X) \to G^*$ by $\theta(\alpha) = \alpha^*$ (assume

$G(f,X)$, $G^*$ have the c/o topology). Clearly $\theta$ is onto and well-defined by definition. If $\alpha_n \to \alpha$, then certainly $\alpha_n^* \to \alpha^*$. Therefore $\theta$ is continuous. If $c \notin f^{-1}(F)$ and $\alpha, \beta \in G(f,X)$, then $\alpha(c), \beta(c) \notin f^{-1}(F)$. Thus, $\alpha^*(\beta^*(c)) = (\alpha \circ \beta)^*(c)$.

4.19. Theorem. $G^* \oplus F \in V(C)$ and $C/\sim \simeq X/F$ where $\sim$ is given by 4.0. with respect to the group $G^* \oplus F$.

Proof: $\sim$ is certainly reflexive and symmetric. Suppose $c_1 \sim c_2$, $c_2 \sim c_3$, and $\varepsilon > 0$ is given. There are $a, b \in C$ and $h, k \in G^* \oplus F$ such that $d(c_1,a) + d(h(a),c_2) < \varepsilon$ and $d(c_2,b) + d(h(b),c_3) < \varepsilon$. If $c_1 \notin f^{-1}(F)$, then we can assume $a \notin f^{-1}(F)$. Now $h = g \circ \alpha$, $g \in G^*$ and $\alpha \in F$. Therefore $\alpha(a) = a$, $h(a) = g(a) \notin f^{-1}(F)$, and $b \notin f^{-1}(F)$. Thus, $c_1 \sim c_2$ (with respect to $G(f,X)$) and in a similar way $c_2 \sim c_3$ (with respect to $G(f,X)$). Therefore there is an $e \in C$ and $g' \in G(f,X)$ such that $d(c_1,e) + d(g'(e),c_3) < \varepsilon$. Again we assume $e \notin f^{-1}(F)$. Thus $g'(e) = (g')^*(e)$. If $c_1 \in f^{-1}(F)$, then $a \in f^{-1}(F)$ and $h(a) \in f^{-1}(F)$. Therefore, $c_2 \in f^{-1}(F)$ and $c_3 \in f^{-1}(F)$. $f^{-1}(F)$ is homogeneous and we have $c_1 \sim c_3$ (with respect to $G^* \oplus F$) as asserted. Hence, $G^* \oplus F$ is transitive and $G^* \oplus F \in V(C)$.

Clearly $G^* \cap F = \{1_C\}$ and $g\alpha = \alpha g$ for each $g \in G^*$ and $\alpha \in F$. Therefore $G^* \oplus F$ is a group.

Consider $p : C \to X/F$ given by $p(c) = \eta(f(c))$, $\eta : X \to X/F$ the quotient map. $G(p,X/F)$ is precisely $G^* \oplus F$ and therefore $C/\sim$ (with respect to $G^* \oplus F$) $\simeq X/F$.

# CHAPTER V

## TOPOLOGICAL PROPERTIES OF STANDARD MAPS AND SOME RELATIONSHIPS BETWEEN X AND G(f,X)

In this section all spaces are compact and metrizable. We will also be considering the function spaces $S(C,X)$ and $G(f,X)$ as well as the group $G(f,X)$. All function spaces are endowed with the compact-open topology.

We will develop some of the basic topological properties possessed by $S(C,X)$ and $G(f,X)$. We will also determine sufficient conditions on $G(f,X)$ which force X to have certain characteristics.

A reasonable starting point is the investigation of the relationship between $S(C,X)$ and $C(C,X)$.

5.0. Theorem. Let M be a compact metric space and $f \in C(C,M)$. If $p \in S(C,M)$, then there is a $g \in S(C,C)$ such that $p \circ g \in S(C,M)$ and $\text{dist}(f,p\circ g) < \varepsilon$ for any $\varepsilon > 0$.

Proof: Given $\varepsilon$, let $\{C_j : 1 \le j \le n\}$ be a decomposition of C satisfying following conditions:

1. Each $C_i$ is both open and closed,
2. $C_i \cap C_j = \phi$, $i \neq j$, and
3. $\text{diam } f(C_j) < \varepsilon$ for each i.

Define $E_j = p^{-1}(f(C_j))$ and $D_j = C \times E_j \times \{1/j\}$. Let $\mathcal{D} = \cup\{D_j : 1 \le j \le n\}$ and note $D_j \simeq \mathcal{D} \simeq C$. If $h \in G(p,M)$, then $h(e_j) = E_j$ for all

j. Construct $\alpha \in H(C,\mathcal{D})$ such that $\alpha(C_j) = D_j$ and define $q \in \alpha C(C,C)$ by $q = \Pi\circ\alpha$ where $\Pi$ is the projection map on the $E_j$ - coordinate. $\Pi \in S(\mathcal{D},C)$. If $\Pi(c_1,e_1,1/j_1) = \Pi(c_2,e_2,1/j_2)$, then $e_1 = e_2$ by definition of $\Pi$. Let $x_n = (c_1,e_1,1/j_1)$ for each n and choose $h \in H(C)$ and $k \in H(\{1/1,1/2,\ldots,1/n\})$ such that $h(c_1) = c_2$ and $k(1/j_1) = 1/j_2$. Then $(h,1_C,k) \in G(\Pi,C)$ and $(h,1_C,k)(x_n) \to (c_2,e_2,1/j_2)$ (equals in fact) as required. In view of 3.6 $\Pi\circ\alpha \in S(C,C)$. Define $g \in C(C,M)$ by $q = p\circ q = (p\circ\Pi)\circ\alpha$. By 3.6 $g \in S(C,M)$ if $p\circ\Pi \in S(\mathcal{D},M)$. But $p\circ\Pi \in S(\mathcal{D},M)$ by 3.3. Hence $g \in S(C,M)$.

Let $x \in C$ and $x \in C_j$ for some j. Consider $g(x) = p(\Pi(\alpha(x)))$. Then $\alpha(x) \in D_j$ and $\Pi(\alpha(x)) \in E_j$. Therefore, $p(\Pi(\alpha(x))) \in f(c_j)$ which yields $\mathrm{dist}(f,p\circ q) < \varepsilon$.

<u>5.1. Corollary</u>. $S(C,M)$ is a dense subset of $C(C,M)$.

<u>Remark</u>: Note that 5.0 actually states that $pS(C,C) \cap S(C,M)$ is a dense subset of $C(C,M)$ for each $p \in S(C,M)$.

<u>5.2. Definition</u>. Let $f \in S(C,M)$ and $K(f)$ the decomposition of C by the point-inverses of f. Define <u>A(f,M)</u> $= \{\alpha \in H(C) : \alpha(D) \in K(f)$ for each $D \in K(f)\}$. It is convenient to think of the elements of $A(f,M)$ ($G(f,M)$) as those elements of $H(C)$ which switch (preserve) the fibers of f.

<u>5.3. Theorem</u>. The following statements are true.

(1) There is a continuous homomorphism from $A(f,M)$ to $H(M)$ with kernel $G(f,M)$.

(2) $A(f,M)$ is a closed subset of $H(C)$.

(3) $G(f,M)$ is a closed (in $H(C)$) normal subgroup of $A(f,M)$.

Proof: (1) Let $h \in A(f,M)$ and define $\alpha(h)$ by $\alpha(h)(m) = f(h(f^{-1}(m)))$. By a well-known theorem (see (3), page 123) $\alpha(h)$ is continuous if $\alpha(h)$ is single-valued. But this is clearly true since h preserves the fiber structure of f. Similarly $[\alpha(h)]^{-1}$ is continuous and therefore $\alpha(h) \in H(M)$.

$\alpha$ is continuous: Let $(F,U)$ be any subbasic open set of $H(M)$. Then $\alpha^{-1}((F,U)) = (f^{-1}(F), f^{-1}(U)) \cap A(f,M)$. Hence $\alpha$ is continuous.

$\alpha$ is a homomorphism: Let $h_1, h_2 \in A(f,M)$. Then $\alpha(h_1 \circ h_2) = f \circ h_1 \circ h_2 \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1})$ since $h_1$ and $h_2$ preserve the fiber structure of f. Thus $\alpha(h_1 \circ h_2) = \alpha(h_1) \circ \alpha(h_2)$ as required.

kernel $\alpha = G(f,M)$: Clearly for each $h \in G(f,M)$ $\alpha(h) = 1_M$. If $\alpha(h) = 1_M$, then by definition $f \circ h \circ f^{-1} = 1_M$. For each $m \in M$ we have $f(h(f^{-1}(m))) = m$, or equivalently, $h(f^{-1}(m)) \subset f^{-1}(m)$. But $h \in A(f,M)$ and hence $h(f^{-1}(m)) = f^{-1}(m)$. Therefore $h \in G(f,M)$ and $G(f,M) =$ kernel $\alpha$.

(2) Let $h \in \overline{A(f,M)}$ and let $h_n \in A(f,M)$ be a sequence such that $h_n \to h$. Let $\alpha$ be the continuous homomorphism defined in (1). We will show that $\alpha(h_n)$ converges to some element k of $H(M)$ and that $\alpha(h) = k$.

Let p and $p^+$ (d and $d^+$) be complete metrics for C and $C(C)$ (M and $C(M)$), respectively. Let $\varepsilon > 0$ be given and choose $\delta$ according to the uniform continuity of f such that $d(f(x),f(y)) < \varepsilon$ for each x and y satisfying $p(x,y) < \delta$. Since $h_n \to h$ there is an integer N such that $n,m > N$ implies $p^+(h_n, h_m) < \delta$. Thus, $d(\alpha(h_n)(y),\alpha(h_m)(y)) = d(f(h_n(f^{-1}(y))), f(h_m(f^{-1}(y)))) < \varepsilon$ provided $n,m > N$, y arbitrary. Hence $\alpha(h_n)$ is a Cauchy sequence in the complete space $C(M)$. Let $\lim \alpha(h_n) = k \in C(M)$. Then $f \circ h_n \circ f^{-1} \to k$ and $f \circ h_n \to k \circ f$. Therefore $f \circ h = k \circ f$. In a similar way $f \circ h_n^{-1} \to f \circ h^{-1} = k' \circ f$, $k' \in C(M)$. Hence

$k \circ k' = 1_M = k' \circ k$. Therefore $k \in H(M)$. $f \circ h \circ f^{-1} = k$ implies $h \in A(f,M)$. We conclude that $A(f,M)$ is closed.

(3) $G(f,M)$ is a closed (in $A(f,M)$) normal subgroup of $A(f,M)$ by virtue of (1). Since $A(f,M)$ is closed in $H(C)$, we have $G(f,M)$ is closed in $H(C)$.

The next step, having established 5.3, is to show $A(f,M) \neq G(f,M)$ for some $f \in S(C,M)$. We do not know if this is the case for all $f \in S(C,M)$. Obviously $G(f,M) = A(f,M)$ for some choices of M. For example, any rigid compact metric space (A space is rigid if it's homeomorphism group is trivial. This type of result can be found in (7)). The following definitions and theorem 5.7 are due to R.D. Anderson (5). We will sketch it's proof for later use.

5.4. Definition. Y is an infinite product space if $Y \simeq \Pi\{Y_i : i \in Z$ where $Y_i \simeq Y$ for all i and Z is the set of integers.

If $g_i \in H(Y_i,Y)$ for each $i \in Z$, then we define the infinite shift $\sigma \in H(\Pi\{Y_i : i \in Z\})$ by $\sigma(\{y_i : i \in Z\}) = \{x_i : i \in Z\}$ where $x_i = g_i^{-1}(g_{i+1}(y_{i+1}))$.

5.5. Definition. A discrete flow is a triple $(Z,Y,\alpha)$ where Z is the additive group of integers, Y is a separable metric space, and $\alpha$ is a map from Z x Y onto Y such that (1) $\alpha(n_1+n_2,y) = \alpha(n_1, \alpha(n_2,y))$, and (2) $\alpha(0,y) = y$ for each $y \in Y$. If $Y' \subset Y$ and $\alpha(Z,Y') = Y'$, then we say that $(Z,Y',\alpha)$ is a subflow of $(Z,Y,\alpha)$.

The flow $(Z,X,\beta)$ is lifted by f to the flow $(Z,Y,\alpha)$ if $f \in C(Y,X)$ and the following diagram commutes.

$$\begin{tikzcd} Z \times Y \arrow[r, "\alpha"] \arrow[d, "(1,f)"'] & Y \arrow[d, "f"] \\ Z \times X \arrow[r, "\beta"'] & X \end{tikzcd}$$

5.6. Definition. The flow $(Z,Y,\alpha)$ is said to be quasi-universal with respect to a class $\chi$ of spaces if for any $X \in \chi$, any flow $(Z,X,\beta)$ can be lifted by a mapping f to a subflow of $(Z,Y,\alpha)$. $(Z,Y,\alpha)$ is universal with respect to $\chi$ if $(Z,Y,\alpha)$ is quasi-universal and f may be specified to be a homeomorphism.

5.7. Theorem. For any infinite product space Y, the infinite shift $\sigma$ generates a universal discrete flow with respect to the class of spaces which can be embedded in Y.

Proof: Let $f_0 : X \to Y$ be an embedding and let $h \in H(X)$. Let $(Z,X,\beta)$ be the discrete flow defined by $\beta(n,x) = h^n(x)$ and let $(Z,Y,\sigma)$ be the discrete flow defined by $\sigma(n,y) = \sigma^n(y)$.

Define $f : X \to Y$ by $f(x) = \{y_i : i \in Z\}$ where $y_i = g_i^{-1}(f_0(h^i(x)))$. Clearly f is an embedding. Set $A = f(X)$. We require $(Z,A,\alpha)$ to be a subflow of $(Z,Y,\sigma)$ and the following diagram to be commutative.

$$\begin{tikzcd} (Z,A,\sigma) \arrow[r, "\sigma"] \arrow[d, "(1,f^{-1})"'] & A \arrow[d, "f^{-1}"] \\ (Z,X,\beta) \arrow[r, "\beta"'] & X \end{tikzcd}$$

Let $f(x) = \{a_i : i \in Z\}$. Then $a_{i+1} = g_{i+1}^{-1}(f_0(h^{i+1}(x)))$, or equivalently, $g_i^{-1}(g_{i+1}(a_{i+1})) = g_i^{-1}(f_0(h^{i+1}(x))) = g_i^{-1}(f_0(h^i(h(x))))$. Hence $\sigma f^{-1} = f^{-1}h$.

The next theorem is a slight generalization of a result by Anderson (5). It is a standard map version of his theorem and therefore relevant to our discussion.

5.8. Theorem. Let M be compact and metrizable. If $h \in H(M)$, then there is a standard map $\phi \in S(C,M)$ and $\alpha \in H(C)$ such that $\phi\circ\alpha = h\circ\phi$.

<u>Proof</u>: Let $(Z,C,\sigma_C)$ be the discrete flow on the Cantor set C generated by $\sigma_C$. Let $g_i$ be the homeomorphisms associated with the infinite product. Similarly, we define $(Z,I^\infty,\sigma_{I^\infty})$ with associated homeomorphisms $g_i' \in H(I_i^\infty,I^\infty)$, where $I^\infty$ is the Hilbert cube.

Let $f_0 \in S(C_0,I_0^\infty)$ and define $f \in C(C,I^\infty)$ by $\Pi_k(f(\{x_i : i \in Z\})) = g_k'^{-1}(g_0'(f_0(g_0^{-1}(g_k(\Pi_k(\{x_i : i \in Z\}))))))$. We assert that $f \in S(C,I^\infty)$ and $\sigma_{I^\infty}\circ f = f\circ\sigma_C$. The latter statement follows since $g_{k-1}'^{-1}(g_k'(\Pi_k(f(\{x_i\}))))$ $= g_{k-1}'^{-1}(g_0'(f_0(g_0^{-1}(g_{k-1}(g_{k-1}^{-1}(g_k(\Pi_k(\{x_i\}))))))))$, or $\sigma_{I^\infty}\circ f = f\circ\sigma_C$. Now, let us assume that $f(\{x_i\}) = f(\{y_i\})$. For each k, $f_0g_0^{-1}g_k \in S(C_k,I_0^\infty)$ by lemma 3.6. Therefore, for each $k \in Z$ there are sequences $\{z_n^k : n \in N\}$ in $C_k$ and $h_n^k \in G(f_0\circ g_0^{-1}\circ g_k,I_0^\infty)$ such that $z_n^k \to x_k$ and $h_n^k(z_n^k) \to y_k$. Define $H_n \in H(C)$ by $H_n = \{h_n^k : k \in Z\}$. Clearly $H_n \in G(f,I^\infty)$ and $H_n(\{z_n^k : k \in Z\})$ converges to $\{y_k : k \in Z\}$. Since f is closed it is an identification and hence standard.

Let $(Z,M,\beta)$ be the flow generated by $h \in H(M)$. By 5.7 $(Z,M,\beta)$ can be embedded as a subflow of $(Z,I^\infty,\sigma)$. Let $(Z,F,\sigma)$ be that subflow. We have

$$\begin{array}{ccc} Z \times F & \xrightarrow{\sigma} & F \\ {\scriptstyle (1,k)}\downarrow & & \downarrow{\scriptstyle k} \\ Z \times M & \xrightarrow[\beta]{} & M \end{array}$$

where $k^{-1} : M \to F$ is a homeomorphism. Set $C' = f^{-1}(F)$ and consider the following commutative diagram.

$$\begin{tikzcd}
Z \times C' \arrow[r, "\sigma_C"] \arrow[d, "{(1,\phi)}"'] & C' \arrow[d, "\phi"] \\
Z \times M \arrow[r, "\beta"'] & M
\end{tikzcd}$$

where $\phi = k(f|_{f^{-1}(F)})$. $C'$ is closed since $M$ is compact. Therefore $C \times C'$ is a Cantor set. Define the flow $(Z, C' \times C, \overline{\sigma}_C)$ by $\overline{\sigma}_C(n,c',c) = (\sigma_C(c',n),c)$. Then

$$\begin{tikzcd}
Z \times (C' \times C) \arrow[r, "\overline{\sigma}_C"] \arrow[d, "{(1,\Pi_{C'})}"'] & C' \times C \arrow[d, "\Pi_{C'}"] \\
Z \times C' \arrow[r, "\sigma_C"] \arrow[d, "{(1,\phi)}"'] & C' \arrow[d, "\phi"] \\
Z \times M \arrow[r, "\beta"'] & M
\end{tikzcd}$$

is a commutative diagram and $\phi = \phi\Pi_{C'} \in S(C' \times C, M)$.

<u>5.9. Lemma</u>. Let $M$ be a compact metric space, $f \in C(C,M)$, $h \in H(M)$, and $K(f)$ the decomposition of $C$ by the point-inverses of $f$. If $f\alpha = hf$ for some $\alpha \in H(C)$, then $\alpha(D) \in K(f)$ for each $D \in K(f)$.

<u>Proof</u>: Let $m \in M$. Then $f(\alpha(f^{-1}(m))) = h(f(f^{-1}(m)) = h(m)$, or $\alpha(f^{-1}(m)) \subset f^{-1}(h(m))$. Therefore, for each $D \in K(f)$ there is a $D' \in K(f)$ such that $\alpha(D) \subset D'$. Similarly, for each $D \in K(f)$ there is a $D' \in K(f)$ such that $\alpha^{-1}(D) \subset D'$. If $\alpha(D) \neq D'$, then there is a $d' \in D'$ such that $d' \notin$

$\alpha(D)$. But $\alpha^{-1}(D') \cap D \neq \phi$. Hence $\alpha(D) = D'$.

We can finally state and prove the promised theorem.

5.10. Theorem. Let M be a non-rigid compact space. Then there is a Cantor set C and standard map f from C onto M such that $A(f,M) \neq G(f,M)$.

Proof: Follows immediately from 5.8 and 5.9.

Chapter five up to this point has dealt only with the properties possessed by $G(f,M)$ and $A(f,M)$ and their relationship. We will now establish conditions on $G(f,M)$ which force M to assume certain topological characteristics.

Let M be a compact metric space and $p \in S(C,M)$. Recall that for $N \subset M$, $S(p,N) = \{\alpha \in G(p,M) : \alpha(x) = x \text{ for each } x \text{ in } C - p^{-1}(N)\}$ is a normal subgroup of $G(p,M)$.

Remark: If N is a closed-open subset of M, then $N \simeq p^{-1}(N)/\sim$ ($\sim$ with respect to $S(p,N)$ as defined in 3.0.). This is an easy consequence of the following argument. Define $\overline{p} : p^{-1}(N) \to N$ by $\overline{p}(x) = p(x)$. For each $\alpha \in G(p,M)$ set

$$\overline{\alpha}(x) = \begin{cases} \alpha(x) & x \in p^{-1}(N) \\ x & x \notin p^{-1}(N) \end{cases}$$

and note that $\overline{\alpha} \in S(p,N)$. Thus, $\overline{p}(x) = \overline{p}(y)$ implies there are sequences $x_n \in p^{-1}(N)$ and $h_n \in G(p,M)$ satisfying $x_n \to x$ and $h_n(x_n) \to y$. Clearly $\overline{h}_n \in (Sp,N)$ and $\overline{h}_n(x_n) \to y$ as required.

A remark due to A.R. Vobach (16) is the following: If $X \cup Y$ is a separation of M, then $G(p,M) = S(p,X) \oplus S(p,Y)$. A converse is true.

5.11. Theorem. Let M be a compact metric space. M is not connected if and only if for each $f \in S(C,M)$ we have

1) $G(f,M) = H \oplus K$ and

2) a separation $C = C_H \cup C_K$ such that $H$ fixes each element of $C_K$ and $K$ each element of $C_H$.

Proof: We denote $C/\sim_G$, $C_H/\sim_H$ and $C_K/\sim_K$ by $C/G$, $C_H/H$ and $C_K/K$, respectively. We use the symbol $\cup_F$ to denote free union. Define $h : C/G \to C_H/H \cup_F C_K/K$ by

$$h([x]_G) = \begin{cases} [x]_H & x \in C_H \\ [x]_K & x \in C_K \end{cases}$$

Let $p_G : C \to C/G$, $p_H : C_H \to C_H/H$ and $p_K: C_K \to C_K/K$ be the quotient maps. Suppose $B$ is a closed subset of $C_H/H \cup_F C_K/K$, say $B = B_H \cup_F B_K$. Then $h^{-1}(B) = p_G(p_H^{-1}(B_H)) \cup p_G(p_k^{-1}(B_K))$ is closed. Clearly $h$ is one-to-one and therefore a homeomorphism. Hence $M$ is not connected.

The converse follows immediately from Vobach's remark since $p^{-1}(X) \cup p^{-1}(Y)$ is a separation of the desired type.

An idea closely related to 5.11 is expressed in the following theorem.

5.12. Theorem. Let $H$ and $K \in V(C)$ be such that there is a separation $C_H \cup C_K = C$ such that $H$ and $K$ fix each element of $C_K$ and $C_H$, respectively. Then the following statements are true.

(1) $G = H \oplus K \in V(C)$

(2) $C/G \simeq C_H/H \cup_F C_K/K$

Proof: (1) Clearly $h \oplus K$ is a subgroup of $H(C)$. In view of 4.5 it suffices to show that $\sim$ (with respect to $H \oplus K$, see 4.0.) is transitive. Suppose $x \sim y$ and $y \sim z$, and let $\varepsilon > 0$ be given. Choose $a, b \in C$ and

$h_1 \cdot k_1$, $h_2 \cdot k_2 \in G$ satisfying $d(x,a) + d(h_1k_1(a),y) < \varepsilon$ and $d(y,b) + d(h_2k_2(b),z) < \varepsilon$. If $x \in C_H$, then without loss of generality we can assume $a \in C_H$. Thus, $h_1k_1(a) = h_1(a)$ which implies $x \sim_H y$ (with respect to H). Similarly, $y \sim_H z$. $H \in V(C)$ implies $x \sim_H z$ and hence $x \sim z$. Therefore, $G \in V(C)$. A similar argument applies if $x \in C_K$. The proof of (2) is contained in 5.11.

The last two theorems in this section deal with contractibility and the fixed-point property.

<u>5.13. Definition</u>. Let $C' \subset C$ and $\mathcal{D}'$, $\mathcal{D}$ decompositions of $C'$ and $C$, respectively. A retraction $r : C \to C'$ is called <u>fiber-preserving</u> if given $D \in \mathcal{D}$ we have $[r(c_1)]_{\mathcal{D}'} = [r(c_2)]_{\mathcal{D}'}$ for each $c_1, c_2 \in D$.

<u>Remark</u>: Note that any closed subset of C is a retract of C.

<u>5.14. Theorem</u>. Let X be a compact metric space embedded in $I^\infty$. Let $\phi \in C(C,I^\infty)$ be continuous and $C' = \phi^{-1}(X)$. Define $\phi' = \phi|_{C'}$ and let $K(\phi)$ and $K(\phi')$ be the decompositions of C and C', respectively, induced by the point-inverses of $\phi$ and $\phi'$ respectively. If there is a fiber-preserving (with respect to $K(\phi)$, $K(\phi')$) retraction $r : C \to C'$, then X is a contractible Peano continuum.

<u>Proof</u>: We identify $C/K(\phi)$ and $C/K(\phi')$ with $I^\infty$ and X, respectively. Consider the following diagram

$$\begin{tikzcd} C/K(\phi) \arrow[r, "\overline{r}"] & C'/K(\phi') \\ C \arrow[u, leftarrow, "\phi"'] \arrow[r, "r"'] & C' \arrow[u, leftarrow, "\phi'"] \end{tikzcd}$$

where $\overline{r}([c]_{K(\phi)}) = \phi' r \phi^{-1}([c]_{K(\phi)})$. $\overline{r}$ is well-defined since r is fiber-preserving and hence is continuous. Clearly $\overline{r}$ is a retraction and thus X is a contractible Peano continuum.

Remark: The interesting question, at least from our point of view, is what happens if we require $\phi$ to be standard? What properties must $G(\phi, I^{\infty})$ possess in order to induce $\overline{r}$?

5.15. Theorem. Let X be a compact metric space. Then X has the f.p.p. (fixed point property) with respect to homeomorphisms if and only if for each $f \in S(C,X)$ and $h \in A(f,X)$ there is at least one $D \in K(f)$ such that $h(D) = D$.

Proof: Suppose X has the f.p.p. with respect to H(X) and let $f \in S(C,X)$. Consider A(f,X) and let $\alpha \in A(f,X)$. As before, theorem 5.3 we have a continuous homomorphism $\psi : A(f,X) \to H(X)$. Then there is an $x \in X$ such that $(\psi(\alpha))(x) = f(\alpha(f^{-1}(x))) = x$. Since $\alpha \in A(f,X)$, we conclude that $\alpha(f^{-1}(x)) = f^{-1}(x)$.

Conversely, let $h \in H(X)$. By 5.10 there is a standard map f and $\alpha \in A(f,X)$ such that $f \circ \alpha \circ f^{-1} = h$. By hypothesis there is a $D \in K(f)$ such that $\alpha(D) = D$. Set $m = f(D)$. Then $h(m) = f(\alpha(f^{-1}(m))) = m$ and X has the f.p.p. with respect to homeomorphisms.

Remark: Again, the question to ask is which property of G(f,X) will produce such behavior in A(f,X)?

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